Noncommutative Dynamics of Random Operators

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We continue our program of unifying general relativity and quantum mechanics in terms of a noncommutative algebra $\mathcal{A}$ on a transformation groupoid $\Gamma = E \times G$ where E is the total space of a principal fibre bundle over spacetime, and G a suitable group acting on Γ. We show that every $a \in \mathcal{A}$ defines a random operator, and we study the dynamics of such operators. In the noncommutative regime, there is no usual time but, on the strength of the Tomita–Takesaki theorem, there exists a one-parameter group of automorphisms of the algebra $\mathcal{A}$ which can be used to define a state dependent dynamics; i.e., the pair $(\mathcal{A}, \varphi)$, where φ is a state on $\mathcal{A}$, is a "dynamic object." Only if certain additional conditions are satisfied, the Connes–Nikodym–Radon theorem can be applied and the dependence on φ disappears. In these cases, the usual unitary quantum mechanical evolution is recovered. We also notice that the same pair $(\mathcal{A}, \varphi)$ defines the so-called free probability calculus, as developed by Voiculescu and others, with the state φ playing the role of the noncommutative probability measure. This shows that in the noncommutative regime dynamics and probability are unified. This also explains probabilistic properties of the usual quantum mechanics.

KEY WORDS: general relativity; quantum mechanics; unification theory; noncommutative dynamics; random operators; free probability.

1. INTRODUCTION

In a series of works (Heller *et al.*, 1997, 2000; Heller and Sasin, 1999) we have formulated a program to unify general relativity and quantum mechanics based on a noncommutative algebra on a transformation groupoid. In (Heller *et al.*, 2004a) we have tested the program by constructing a simplified (but still mathematically interesting) model and computing many of its details, and in (Heller *et al.*, 2004b) we have discussed its observables with a special emphasis on the position and momentum observables. In the present work, we study its dynamic and probabilistic aspects.

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Let us first, for the reader's convenience, outline the main architectonic properties of our model. We construct a transformation groupoid in the following way. Let $\tilde{E}$ be a differential manifold and $\tilde{G}$ a group acting smoothly and freely on $\tilde{E}$. We thus have the bundle $(\tilde{E}, \pi_M, M = \tilde{E}/\tilde{G})$, and we can think of it as of the frame bundle, with $\tilde{G}$ the Lorentz group, over spacetime M. To simplify our construction, we choose a finite subgroup G of $\tilde{G}$ and a cross section $S : M \to \tilde{E}$ of the above bundle (it need not be continuous). Then we define $E = \cup_{x \in M} S(x)G$. The free action of G (to the right) on E, defines the transformation groupoid structure on the Cartesian product $\Gamma = E \times G$ [for details see (Heller *et al.*, 2004a)]. The choice of the cross section $S : M \to \tilde{E}$ can be regarded as the choice of a gauge for our model.

The elements of the groupoid Γ represent symmetry operations of the model. The noncommutative algebra $\mathcal{A} = C^\infty(\Gamma, \mathbf{C})$ of smooth complex valued functions on Γ (if necessary, we shall assume that they vanish at infinity) with convolution as multiplication is an algebraic counterpart of this symmetry space. In the previous works, we have reconstructed geometry of the groupoid $\Gamma = E \times G$ in terms of this algebra. By projecting the full geometry onto the E-direction we recover the usual spacetime geometry and, consequently, the standard general relativity. The regular representation $\pi_p : \mathcal{A} \to \text{End}(\mathcal{H}_p)$ of the groupoid algebra $\mathcal{A}$ in a Hilbert space $\mathcal{H}_p$, for $p \in E$, gives the G-component of the model which can be considered as its quantum sector.

In the present paper, we show that every bounded function $a \in \mathcal{A}$ defines a random operator (Section 2), and we study the dynamics of these operators (Section 3). This is not a trivial task. Noncommutative spaces are nonlocal entities. In general, the concepts such as that of point and its neighborhood are meaningless in them. Therefore, in the noncommutative setting the concept of the usual "coordinate time" is not applicable, and the question concerning the existence of dynamics arises. However, as shown by Alain Connes (1994) the algebra $\mathcal{M}$ admits, on the strength of the Tomita–Takesaki theorem, a one-parameter group of automorphisms of $\mathcal{M}$ (called the *modular group*), and this group can be used to define a "modular dynamics" (Connes and Rovelli, 1994). But, strangely enough, this dynamics depends on the state φ on the algebra $\mathcal{M}$, and only if certain additional conditions are satisfied, the dependence on φ disappears, and one recovers the usual unitary evolution, well known from quantum mechanics (Section 4).

In Section 5, we briefly recall the noncommutative probability calculus (called also *free probability calculus*) as it was introduced by Voiculescu (1985), and developed by others (Voiculescu *et al.*, 1992; Biane, 1998). Such a probability is defined as the pair $(\mathcal{M}, \varphi)$ where $\mathcal{M}$ is an associative algebra (with unity), and φ is a state on $\mathcal{M}$, i.e., a positive linear functional on $\mathcal{M}$ such that $\varphi(1) = \mathbf{1}$. We can think of φ as of a probability measure on $\mathcal{M}$. We thus see that the pair $(\mathcal{M}, \varphi)$ is both the "dynamic object" and the "probabilistic object." It follows that, in our model, every dynamics is probabilistic (in the generalized sense), and every

(generalized) probability has a dynamic aspect. This important property of the noncommutative regime, supposedly governing the fundamental level of physics, is inherited by the quantum sector of our model. In this way, the probabilistic character of the standard quantum mechanical (unitary) evolution is explained.

Finally, in Section 6, we briefly comment on the obtained results.

2. ALGEBRA OF RANDOM OPERATORS

Let $\Gamma = E \times G$ be the groupoid described in the Introduction. In this paper, unless explicitly stated otherwise, G will always be a finite group. We consider the algebra $\mathcal{A} = C^\infty(\Gamma, \mathbf{C})$ of smooth complex valued functions on Γ with the convolution as multiplication. If $a, b \in \mathcal{A}$, the convolution is defined as

$$(a * b)(\gamma) = \sum_{\gamma_1 \in \Gamma_{d\gamma}} a\big(\gamma \circ \gamma_1^{-1}\big) g(\gamma_1)$$

where $\gamma, \gamma_1 \in \Gamma$, and $\Gamma_{d\gamma}$ denotes the fiber of the groupoid Γ over $d(\gamma) = d(p, g) = p \in E$ with $g \in G$ [for details see (Heller *et al.*, 2004a)].

Every $a \in \mathcal{A}$ generates a random operator $r_a = (\pi_p(a))_{p\in E}$. It acts on a collection $\{\mathcal{H}^p\}_{p\in E}$ of Hilbert spaces $\mathcal{H}^p = L^2(\Gamma^p)$. Here Γ^p denotes the fiber of Γ consisting of all its elements "ending at" $p \in E$. Every operator $\pi_p(a)$ is a bounded operator on $\mathcal{H}^p$.

An operator r_a to be random must satisfy the following conditions:

(1) If $\xi_p, \eta_p \in \mathcal{H}^p$ then the function $E \to \mathbf{C}$ given by

$$E \ni p \mapsto ((r_a)_p \xi_p, \eta_p),$$

for $a \in \mathcal{A}$, is measurable in the usual sense (i.e., with respect to the standard manifold measure on E). In our case this condition is always satisfied.

(2) The operator r_a must be *bounded*, i.e., $||r_a|| < \infty$ where

$$\|r_a\| = \text{ess sup} \|\pi_p(a)\|.$$

Here "ess sup" denotes essential supremum, i.e., supremum modulo zero measure sets. Let us notice that if, in our case, a is a bounded function, condition (2) is satisfied, and if a is continuous, condition (1) is satisfied.

Let $\mathcal{M}$ be the $*$-algebra of equivalence classes (modulo equality almost everywhere) of bounded random operators $(A_p)_{p\in E}$ equipped with the following operations:

1. $(A + B)_p = A_p + B_p$,
2. $(A^*)_p = (A_p)^*$,
3. $(A \cdot B)_p = A_p \cdot B_p$,

$A, B \in \mathcal{M}$, $p \in E$. The well known result is that $\mathcal{M}$ forms a von Neumann algebra, i.e., $\mathcal{M} = \mathcal{M}''$ where $\mathcal{M}''$ denotes the double commutant of $\mathcal{M}$ (Connes,

1994, p. 52). This result clearly applies to our case, i.e., random operators r_a defined above generate a von Neumann algebra. We will call it the *von Neumann algebra of the groupoid* Γ.

In the matrix representation we have (Heller *et al.*, 2004a)

$$L^\infty(\Gamma, \mathbf{C}) \simeq L^\infty(M) \otimes M_{n \times n}(\mathbf{C}).$$

In this representation, the von Neumann algebra of random operators assumes the form

$$\mathcal{M} \simeq L^\infty(E) \otimes M_{n \times n}(\mathbf{C}) \simeq M_{n \times n}(L^\infty(E)).$$

Let us now recall some terminology. Let φ be a positive linear functional on a von Neumann algebra $\mathcal{M}$: φ is said to be *faithful* if $0 \neq x \in \mathcal{M}$ implies $\varphi(x) > 0$; φ is said to be *normal* if $\varphi(x) = \sup\varphi(x_i)$ provided x is the supremum of a monotonically increasing net $\{x_i\}$ in the collection of positive operators in $\mathcal{M}$; φ is called *tracial* if $\varphi(x^*x) = \varphi(xx^*)$ for every $x \in \mathcal{M}$; φ is said to be a *state* if it is positive and normed to unity.

A von Neumann algebra $\mathcal{M}$ is called *finite* if it admits a faithful, normal and tracial state.

In our case, continuous functionals on $\mathcal{M}$ are of the following form

$$\varphi(r_a) = \int \mathrm{Tr}(r_a(p)\rho(p)) \, d\mu_E(p)$$

for $r_a \in \mathcal{M}$, where $\rho \in L^1(E) \otimes M_{n \times n}(\mathbf{C}) \simeq M_{n \times n}(L^1(E))$ or, equivalently, with the dependence on $x \in M$ clearly displayed

$$\varphi(r_a) = \int_M \sum_{g \in G} \mathrm{Tr}(r(s(x) \cdot g) \cdot \rho(s(x) \cdot g)) \, d\mu(x).$$

It is obvious that there exists a tracial functional of this form, namely the one with $\rho(p) = f(x)\mathbf{1}$ where $f \in L^1(M)$, $x = \pi_{M(p)}$

For $\varphi(r_a)$ to be positive, $\rho(p)$ must be a positive matrix, i.e., having all its eigenvalues non-negative, for almost all $p \in E$. If all eigenvalues of $\rho(p)$ are positive, the state is faithful.

Let us define the normalization: if $r_a(p) = \mathbf{1}$, for every $p \in E$, then

$$\varphi(r_a) = \int_M \sum_{g \in G} \mathrm{Tr}(\rho(s(x) \cdot g)) \, d\mu(x) = 1.$$

Proposition 1.   *The von Neumann algebra $\mathcal{M}$ of the groupoid Γ is finite.*

Proof: Let us choose $\rho(s(x) \cdot g) = f(x) \cdot \mathbf{1}$ where $f \in L^1(M)$, and $f > 0$. φ is clearly positive and faithful. Then normalization condition reduces to the following

formula

$$\varphi(r_a) = \int_M n f(x) \, d\mu(x) = 1,$$

where n is the rank of the group G. Therefore, φ is a state. It is also a normal state since every fiber in Γ is finite, and the normality is a simple consequence of the Lebesgue majorized convergence theorem. □

3. EVOLUTION OF RANDOM OPERATORS

Now, we define the Hamiltonian $H_p^\varphi = \mathrm{Log}\rho_p^\varphi$, and the Tomita–Takesaki theorem gives us the evolution of random operators dependent on the state φ in terms of the one-parameter group of automorphisms σ_s^φ, called *modular group* (Connes, 1994, pp. 43–44, 496–470)

$$\sigma_s^\varphi(r_a(p)) = e^{isH_p^\varphi} r_a(p) e^{-isH_p^\varphi} \tag{1}$$

for every $p \in E$.

Eq. (1) can also be written in the form

$$i\hbar \frac{d}{ds}|_{s=0}\sigma_s^\varphi(r_a(p)) = \left[r_a(p), H_p^\varphi\right]. \tag{2}$$

This equation describes the state dependent evolution of random operators with respect to the parameter $s \in \mathbf{R}$ of the modular group.

Our aim is now to obtain the state independent evolution by applying to our case the construction based on the Connes–Nicodym–Radon theorem (Sunder, 1987, p. 74). Let us first recall some concepts involved in this construction. Let $\mathrm{Aut}\mathcal{M}$ be the group of all automorphisms of an algebra $\mathcal{M}$, and $\lambda \in \mathrm{Aut}\mathcal{M}$. An automorphism λ is said to be *inner* if there exists an element $u \in \mathcal{U}$, where $\mathcal{U} = \{u \in \mathcal{M} : uu^* = u^*u = 1\}$ is the unitary group of the algebra $\mathcal{M}$, such that

$$\lambda(b) = ubu^*$$

for every $b \in \mathcal{M}$. Let $\mathrm{Inn}\mathcal{M}$ denote the group of inner automorphisms of $\mathcal{M}$. We define two automorphisms λ_1 and λ_2 to be *inner equivalent* if

$$\lambda_1(b) = u\lambda_2(b)u^*,$$

for every $b \in \mathcal{M}$, and the group $\mathrm{Out}\mathcal{M}$ of *outer automorphism* as

$$\mathrm{Out}\mathcal{M} := \mathrm{Aut}\mathcal{M}/\mathrm{Inn}\mathcal{M}.$$

Let $\sigma_s^\varphi, \sigma_s^\psi \in \mathrm{Aut}\mathcal{M}$ for a fixed $t \in \mathbf{R}$, and let us further assume that there exists the unitary element $u \in \mathcal{U}$ such that

$$\sigma_s^\psi = u\sigma_s^\varphi u^*.$$

Hence,

$$[\sigma_s^\psi] = [\sigma_s^\varphi]$$

where square brackets denote the equivalence class of a given automorphism. If we define the canonical homomorphism

$$\delta : \mathbf{R} \to \mathrm{Out}\mathcal{M}$$

by

$$\delta(s) = [\sigma_s^\varphi],$$

we obtain the modular group which is now state independent.

In our case, we clearly have the state dependent evolution as described by Eq. (1). Could it be made state independent by the above procedure? Eq. (1) implies that $\sigma_s^\varphi \in \mathrm{Inn}\mathcal{M}$, and consequently $\delta(s) = \mathbf{1}$. This means that the one-parameter group σ_s independent of state is trivial.

This can also be deduced from the Dixmier–Takesaki theorem (Connes, 1994, p. 470). Let us define

$$S(\mathcal{M}) = \{S_0 \in \mathbf{R} : \sigma_{S_0}^\varphi \in \mathrm{Inn}\mathcal{M}\}.$$

The Dixmier–Takesaki theorem says that $S(\mathcal{M}) = \mathbf{R}$ if and only if the algebra $\mathcal{M}$ is finite (or semifinite, if the theorem is formulated for weights rather than for states, see below). And, as we know from the previous section, this is indeed the case.

The above result means that every σ_s^φ is unitary equivalent to id_s. In other words, the state independent time does exist, but nothing changes in it. This fact is clearly the consequence of the oversimplified character of our model; in particular, of the fact that the group G is finite.

4. QUANTUM AND CLASSICAL DYNAMICS

So far we have shown that on the fundamental (noncommutative) level we have a state dependent "modular dynamics" which (at least in more realistic models) can be made state independent. Now, the questions arise: what do we get of this dynamics, if we go to the quantum sector and the spacetime sector of our model, respectively?

To answer the first of these questions, let us restrict the von Neumann algebra $\mathcal{M}$ to its subalgebra

$$\mathcal{M}_G = \{f \circ pr_G : f \in \mathbf{C}^G\}$$

where $pr_G : \Gamma \to G$ is the obvious projection. For every $a \in \mathcal{M}_G$, the random operator $r_a = (\pi_p(a))_{p \in E}$ is a family of operators which can be identified with each other (on the strength of the natural isomorphism $\Gamma_p \simeq G$). Therefore, any

such r_a is a family projectible to a single operator on $\mathcal{H}_G = L^2(G)$. The operator on $\mathcal{H}_G$ to which r_a projects will be denoted by a_G. Let us notice that it is not a random operator.

For $a_G \in \text{End}\mathcal{H}_G$, Eq. (2) assumes the form

$$i\hbar \frac{d}{ds}|_{s=0}(\sigma_s^\varphi(a_G)) = [a_G, H^\varphi].$$

The only difference between this equation and the Heisenberg equation, well known from quantum mechanics, is that this equation depends on the state φ. But even this difference disappears for more realistic models in which Connes–Nikodym–Radon theorem gives the state independent modular evolution. In these cases, the standard quantum mechanical dynamics is recovered.

Now, let us turn to the question of what do we obtain by going to the spacetime (macroscopic) sector of our model. In (Heller *et al.*, 2004a) we have shown that the answer can be obtained by the averaging procedure. Let us consider the von Neumann algebra $\mathcal{M}$ in its matrix representation, and let M_a be a matrix corresponding to the function a. Then by averaging of M_a we understand

$$\langle M_a(p)\rangle = \frac{1}{|G|}\text{Tr}M_a$$

where $|G|$ denotes the rank of G.

In (Heller *et al.*, 2004b) we have proved that

$$\pi_{pg}(a) = L_g\pi_p(a)L_{g^{-1}}$$

where L_g denotes the left translation by $g \in G$. By applying the trace operation to both sides of this equality we obtain

$$\text{Tr}(\pi_{pg}(a)) = \text{Tr}(\pi_p(a)),$$

i.e., the averaging operation gives a function on M, and Eq. (1) reduces to

$$\langle\sigma_s^\varphi(r_a(p))\rangle = \langle r_a(p)\rangle.$$

We see that the dependence on φ has disappeared. This equation shows that the modular dynamics (with respect to the parameter s) is a quantum phenomenon which is not directly visible from the spacetime perspective. The "modular time" s is related to the usual "coordinate time" t by the dependence on $p = (x_0, x_1, x_2, x_3, \delta_0, \delta_1, \delta_2, \delta_3) \in E$.

5. DYNAMICS AND PROBABILITY

The fact that dynamics in our model is given in terms of random operators discloses a link between dynamics and probability. This link goes much further.

If X is a compact topological space then there is a strict correspondence between finite Borel measures on X and linear forms on the Banach space $C(X)$ of continuous functions on X with the norm: $\|f\| = \sup|f(x)|$, $f \in C(X)$, $x \in X$. Instead of considering the measure space (X, m), where m is a finite Borel measure on X, we can, equivalently, consider the Banach algebra $C(X)$ together with a distinguished linear form φ on it, i.e., the pair $(C(X), \varphi)$. If, additionally, we impose on φ a suitable normalization condition, this pair will be a functional counterpart of the probability space. This is the starting point of a generalization to the noncommutative concept of probability. In place of the Banach algebra $C(X)$ we consider any associative, not necessarily commutative, unital algebra $\mathcal{A}$. For generality reasons we assume that it is a complex valued algebra. Let further φ be a linear (complex valued) form on $\mathcal{A}$. If it is a noncommutative algebra, the pair $(\mathcal{A}, \varphi)$ is called the *noncommutative probability space*. Noncommutative probability is also called *free probability* (Voiculescu *et al*., 1992; Biane, 1998).

However, the above formulated noncommutative probability is too general for practical purposes. Some additional conditions are required. Also at this stage motivations come from the commutative case. Let $\mathcal{H}$ be a separable Hilbert space, and T a bounded self-adjoint operator on $\mathcal{H}$. It can be shown that

1. There exists the unique (up to equivalence) measure on the interval $I = [-\|T\|, \|T\|]$ such that

$$f(T) = 0 \quad \Leftrightarrow \quad \int |f| d\mu = 0,$$

2. The algebra M of operators on $\mathcal{H}$ having the form $f(T)$, for some bounded Borel function f is a von Neumann algebra (generated by T).

M is a commutative von Neumann algebra naturally isomorphic to the algebra $L^\infty(I, \mu)$ of bounded measurable functions (modulo equality almost everywhere) on the interval I.

In the view of the above, it is natural to regard the theory of von Neumann algebras as a noncommutative counterpart of the measure theory, and to agree for the following definition. The *noncommutative probability space* is a pair $(\mathcal{M}, \varphi)$ where $\mathcal{M}$ is a von Neumann algebra and φ a faithful and normal state on $\mathcal{M}$ (Biane, 1998, Sec. 4). In contrast to the commutative case in which there is only one interesting measure (the Lebesgue measure), the noncommutative case exhibits the great richness and complexity.

As we have seen in Section 3, the pair $(\mathcal{M}, \varphi)$ is a *dynamical object* in the sense that it determines the modular evolution dependent on the state φ. But if φ satisfies certain natural (and, in general, easy to satisfy) conditions, the same pair is a "probabilistic object." Therefore, every such dynamics is probabilistic, and every such probabilistic space has a dynamic aspect. Two structures, which in the standard mathematics were independent of each other, now are unified.

The state φ plays now the role of the probability measure. It is a remarkable fact that it also determines the dynamical regime. If we change from the probability measure φ to the probability measure ψ, then we automatically go from the dynamic regime σ_s^φ to the dynamic regime σ_s^ψ. Only when we succeed in obtaining the nontrivial, state independent evolution $\delta : \mathbf{R} \to \text{Out}\mathcal{M}$, we get the unique unitary probabilistic dynamics typical for quantum mechanics (as described by the Heisenberg equation). Let us notice, however, that the state φ can be interpreted as an expectation value. Therefore, for two (state dependent) inner equivalent modular evolutions this expectation value is the same, i.e., state independent (at least for tracial states).

6. COMMENTS

It is interesting to notice that both dynamics and probability are, from the very beginning, strictly connected with unitarity. Both dynamics and probability are implemented by a von Neumann algebra which can be defined as an algebra of operators on a Hilbert space that are invariant with respect to the group of unitary transformations. This beautifully harmonizes with the fact well known from the standard quantum mechanics that unitarity is closely related to the probabilistic evolution of quantum systems.

To the physicist it might seem astonishing that the modular evolution is originally dependent on a state of the considered system. In fact, it was Carlo Rovelli who proposed a quantum mechanical model with a state dependent time (Rovelli, 1993) and, together with Alain Connes, tried to extend this concept to generally covariant theories, by making the time flow depending on the thermal state of the system (Connes and Rovelli, 1994). Our approach is more radical. We closely follow the conclusions of the Tomita–Takesaki theorem, and assume that on the fundamental level of physics dynamics is indeed state dependent, and only when we move to lower energy levels, the von Neumann algebra $\mathcal{M}$ becomes more "coarse" (in the sense that Aut$\mathcal{M}$ can be replaced by Out$\mathcal{M}$), the Connes–Nicodym–Radon theorem can be applied and time independent dynamics emerges.

In our model this evolution is trivial but, as we have seen, this follows from the simplified character of the model. The Sunder's theorem (Sunder, 1987, p. 88) gives us even more information on the nonexistence of state independent change in our model. The theorem is formulated for weights rather than for states, but it *a fortiori* applies to states. A von Neumann algebra is said to be *semifinite* if it admits a faithful, normal and tracial weight [for definitions see (Sunder, 1987, p. 52)]. Roughly speaking a weight φ on $\mathcal{M}$ is *semifinite* if there are sufficiently many elements of $\mathcal{M}$ at which φ assumes a finite value. The theorem asserts that the following conditions for a von Neumann algebra $\mathcal{M}$ are equivalent:

(i) $\mathcal{M}$ is semifinite;

(ii) σ_t^φ is inner for some faithful, normal and semifinite weight φ on $\mathcal{M}$;

(iii) σ_t^φ is inner for every such weight.

In order to have a nontrivial state independent evolution the von Neumann algebra $\mathcal{M}$ cannot be semifinite. This can be obtained by using in our model a locally compact group. In such a case, one cannot integrate along G to reduce the density $\rho(s(x) \cdot g)$ to the form $f(x) \cdot \mathbf{1}$, and the algebra $\mathcal{M}$ could be nonsemifinite, even if G is a unimodular group.

Our model has disclosed quite unexpected connection between noncommutative dynamics and noncommutative probability. The pair $(\mathcal{M}, \varphi)$ is both the "dynamic object" and the "probabilistic object." This fact throws some light onto the "strange" dependence of the dynamics of random operators on the state φ. The state φ is also a probability measure: if we switch to another state, we switch to another probability measure, and it seems rather natural that together with the change of the probability measure, the dynamical regime of random operators changes as well. Two concepts—dynamics and probability—that are separate in the usual circumstances, in the noncommutative domain turn out to be but two aspects of the same mathematical structure.

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